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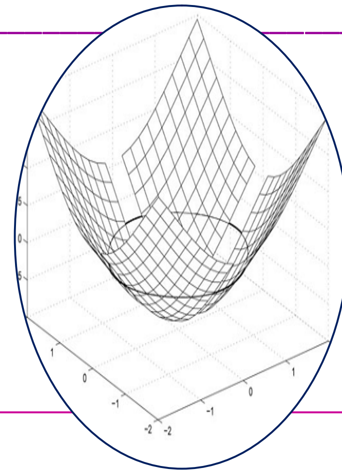

ASYMPTOTIC BEHAVIOUR OF TOEPLITZ MATRICES

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ABSTRACT :

Varies properties of a real symmetric Toeplitz matrix with elements are reviewed here. Matrices of this kind often arise in applications in statistics econometrics, psychometrics, structural engineering, multichannel filtering, reflection seismology etc. and it is desirable to have techniques which exploit their special structure Possible application of the results related to their inverse, determinant and eigenvalue problem are suggested. We started with relevant definitions and a prerequisite and proceed to a discussion of the asymptotic eigenvalue, product and inverse behaviour of matrices. The major use of the theorem of this paper is the relate the asymptotic behaviour of a sequence of complicated matrices to that of a simpler asymptotically equivalent sequence of matrices.

Keywords : *statistics econometrics, psychometrics, structural engineering.*

1. INTRODUCTION

A Toeplitz matrix is an n × n matrix $T_n = [t_{k,j}\,;\, k, j = 0, 1, \ldots, n-1]$ where $t_{k,j} = t_{k-j}$, i.e., a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & & \\ t_2 & t_1 & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{n-1} & & & \cdots & t_0 \end{bmatrix}. \tag{1.1}$$

Such matrices arise in many applications. For example, suppose that

$$x = (x_0, x_1, \ldots, x_{n-1})' = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

is a column vector (the prime denotes transpose) denoting an “input” and that t_k is zero for $k < 0$. Then the vector

$$y = T_n x = \begin{bmatrix} t_0 & 0 & 0 & \dots & 0 \\ t_1 & t_0 & 0 & & \\ t_2 & t_1 & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{n-1} & & & \dots & t_0 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} x_0 t_0 \\ t_1 x_0 + t_0 x_1 \\ \sum_{i=0}^{2} t_{2-i} x_i \\ \vdots \\ \sum_{i=0}^{n-1} t_{n-1-i} x_i \end{pmatrix}$$

with entries

$$yk = \sum_{i=0}^{k} t_{k-i} x_i$$

represents the output of the discrete time causal time-invariant filter h with “impulse response” t_k . Equivalently, this is a matrix and vector formulation of a discrete-time convolution of a discrete time input with a discrete time filter.

A common special case of Toeplitz matrices — which will result in significant simplification and play a fundamental role in developing more general results — results when every row of the matrix is a right cyclic shift of the row above it so that $t_k = t_{-(n-k)} = t_{k-n}$ for $k = 1, 2, \dots, n-1$. In this case the picture becomes

$$C_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(n-1)} \\ t_{-(n-1)} & t_0 & t_{-1} & & \\ t_{-(n-2)} & t_{-(n-1)} & t_0 & & \vdots \\ \vdots & & \ddots & & \\ t_{n-1} & t_{-2} & \dots & t_0 & \end{bmatrix}. \qquad (1.2)$$

A matrix of this form is called a circulant matrix. Circulant matrices arise, for example, in applications involving the discrete Fourier trans-form (DFT) and the study of cyclic codes for error correction.

A great deal is known about the behavior of Toeplitz matrices — the most common and complete references being Grenander and Szeg¨o [16] and Widom [33]. A more recent text devoted to the subject is B¨ottcher and Silbermann [5]. Unfortunately, however, the necessary level of mathematical sophistication for understanding reference [16] is frequently beyond that of one species of applied mathematician for whom the theory can be quite useful but is relatively little understood. This caste consists of engineers doing relatively mathematical (for an engineering background) work in any of the areas mentioned. This apparent dilemma provides the motivation for attempting a tutorial introduction on Toeplitz matrices that proves the essential theorems using the simplest possible and most intuitive mathematics. Some simple and fundamental methods that are deeply buried (at least to the untrained mathematician) in [16] are here made explicit.

The most famous and arguably the most important result describing Toeplitz matrices is Szeg¨o's theorem for sequences of Toeplitz matrices $\{T_n\}$ which deals with the behavior of the eigenvalues as n goes to infinity. A complex scalar α is an eigenvalue of a matrix A if there is a nonzero vector x such that

$$Ax = \alpha x, \tag{1.3}$$

in which case we say that x is a (right) eigenvector of A. If A is Hermitian, that is, if A* = A, where the asterisk denotes conjugate transpose, then the eigenvalues of the matrix are real and hence $\alpha^* = \alpha$, where the asterisk denotes the conjugate in the case of a complex scalar. When this is the case we assume that the eigenvalues $\{\alpha_i\}$ are ordered in a nondecreasing manner so that $\alpha_0 \geq \alpha_1 \geq \alpha_2 \ldots\ldots$ This eases the approximation of sums by integrals and entails no loss of generality. Szeg¨o's theorem deals with the asymptotic behavior of the eigenvalues $\{\tau_{n,i;} i = 0, 1, \ldots, n-1\}$ of a sequence of Hermitian Toeplitz matrices $T_n = [t_{k-j}; k, j = 0, 1, 2, \ldots, n-1]$. The theorem requires that several technical conditions be satisfied, including the existence of the Fourier series with coefficients t_k related to each other by

$$f(\lambda) = \sum_{k=-\infty}^{\infty} t_k e^{ik\lambda}; \lambda \in [0, 2\pi] \tag{1.4}$$

$$t_k = \frac{1}{2\pi}\int_0^{2\pi} f(\lambda) e^{ik\lambda} d\lambda. \tag{1.5}$$

Thus the sequence $\{t_k\}$ determines the function f and vice versa, hence the sequence of matrices is often denoted as $T_n(f)^*$. If $T_n(f)$ is Hermitian, that is, if $T_n(f)^* = T_n(f)$, then $t_{-k} = t_k^*$ and f is real-valued.

Under suitable assumptions the Szeg¨o theorem states that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} F(\tau_{n,k}) = \frac{1}{2\pi}\int_0^{2\pi} F(f(\lambda)) d\lambda. \tag{1.6}$$

for any function F that is continuous on the range of f. Thus, for example, choosing $F(x) = x$ results in

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \tau_{n,k} = \frac{1}{2\pi}\int_0^{2\pi} f(\lambda) d\lambda. \tag{1.7}$$

so that the arithmetic mean of the eigenvalues of $T_n(f)$ converges to the integral of f. The trace $Tr(A)$ of a matrix A is the sum of its diagonal elements, which in turn from linear algebra is the sum of the eigenvalues of A if the matrix A is Hermitian. Thus (1.7) implies that

$$\lim_{n\to\infty} \frac{1}{n} Tr(T_n(f)) = \frac{1}{2\pi}\int_0^{2\pi} f(\lambda) d\lambda. \tag{1.8}$$

Similarly, for any power s

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln \tau_{n,i} = \frac{1}{2\pi}\int_0^{2\pi} \ln f(\lambda) d\lambda. \tag{1.9}$$

If f is real and such that the eigenvalues $\tau_{n,k} \geq m > 0$ for all n, k, then $F(x) = \text{In } x$ is a continuous function on [m, ∞) and the Szeg¨o theorem can be applied to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\ln\tau_{n,i}=\frac{1}{2\pi}\int_0^{2\pi}\ln f(\lambda)\,d\lambda. \tag{1.10}$$

From linear algebra, however, the determinant of a matrix $T_n(f)$ is given by the product of its eigenvalues,

$$\det(T_n(f))=\prod_{i=0}^{n-1}\tau_{n,i}$$

so that (1.10) becomes

$$\lim_{n\to\infty}\ln\det(T_n(f))^{1/n}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln\tau_{n,i}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\ln f(\lambda)\,d\lambda. \tag{1.11}$$

As we shall later see, if f has a lower bound m > 0, than indeed all the eigenvalues will share the lower bound and the above derivation applies. Determinants of Toeplitz matrices are called Toeplitz determinants and (1.11) describes their limiting behavior.

The Asymptotic Behaviour of Matrices

Any complex matrix A can be written as

$$A = U R U^*, \tag{2.1}$$

where the asterisk ∗ denotes conjugate transpose, U is unitary, i.e., $U^{-1} = U^*$, and $R = \{r_{k,j}\}$ is an upper triangular matrix ([18], p. 79). The eigenvalues of A are the principal diagonal elements of R. If A is normal, i.e., if $A^*A = AA^*$, then R is a diagonal matrix, which we denote as $R = diag(\alpha_k;\ k = 0, 1, \ldots, n-1)$ or, more simply, $R = diag(\alpha_k)$. If A is Hermitian, then it is also normal and its eigenvalues are real.

A matrix A is nonnegative definite if $x^*Ax \geq 0$ for all nonzero vectors x. The matrix is positive definite if the inequality is strict for all nonzero vectors x. (Some books refer to these properties as positive definite and strictly positive definite, respectively.) If a Hermitian matrix is nonnegative definite, then its eigenvalues are all nonnegative. If the matrix is positive definite, then the eigenvalues are all (strictly) positive.

The extreme values of the eigenvalues of a Hermitian matrix H can be characterized in terms of the Rayleigh quotient $R_H(x)$ of the matrix and a complex-valued vector x defined by

$$R_H(x) = (x^*Hx)/(x^*x). \tag{2.2}$$

As the result is both important and simple to prove, we state and prove it formally. The result will be useful in specifying the interval containing the eigenvalues of a Hermitian matrix.

Usually in books on matrix theory it is proved as a corollary to the variational description of eigenvalues given by the Courant-Fischer theorem (see, e.g., [18], p. 116, for the case of real symmetric matrices), but the following result is easily demonstrated directly.

Lemma 2.1. Given a Hermitian matrix H, let η_M and η_m be the maximum and minimum eigenvalues of H, respectively. Then

$$\eta_m=\min_x R_H(x)=\min_{z:z^*z=1} z^*Hz \tag{2.3}$$

$$\eta_m = \max_x R_H(x) = \max_{z:z^*z=1} z^* Hz \tag{2.4}$$

Proof. Suppose that e_m and e_M are eigenvectors corresponding to the minimum and maximum eigenvalues η_m and η_M , respectively. Then $R_H(e_m) = \eta_m$ and $R_H(e_M) = \eta_M$ and therefore

$$\eta_m \geq \min_x R_H(x) \tag{2.5}$$

$$\eta_M \geq \max_x R_H(x) \tag{2.6}$$

Since H is Hermitian we can write $H = UAU^*$, where U is unitary and A is the diagonal matrix of the eigenvalues η_k , and therefore

$$\frac{x^* Hx}{x^* H} = \frac{x^* UAU^* x}{x^* x}$$

$$= \frac{y^* Ay}{y^* y} = \frac{\sum_{k=1}^{n} |y_k|^2 \eta_k}{\sum_{k=1}^{n} |y_k|^2},$$

where $y = U^*x$ and we have taken advantage of the fact that U is unitary so that $x^*x = y^*y$. But for all vectors y, this ratio is bound below by η_m and above by η_M and hence for all vectors x

$$\eta_m \leq R_H(x) \leq \eta_M \tag{2.7}$$

which with (2.5–2.6) completes the proof of the left-hand equalities of the lemma. The right-hand equalities are easily seen to hold since if x minimizes (maximizes) the Rayleigh quotient, then the normalized vector x/x^*x satisfies the constraint of the minimization (maximization) to the right, hence the minimum (maximum) of the Rayleigh quotion must be bigger (smaller) than the constrained minimum (maximum) to the right. Conversely, if x achieves the rightmost optimization, then the same x yields a Rayleigh quotient of the the same optimum value.

The following lemma is useful when studying non-Hermitian ma-trices and products of Hermitian matrices. First note that if A is an arbitrary complex matrix, then the matrix A^*A is both Hermitian and nonnegative definite. It is Hermitian because $(A^*A)^* = A^*A$ and it is nonnegative definite since if for any complex vector x we define the complex vector $y = Ax$, then

$$x^*\left(A^*A\right)x = y^*y = \sum_{k=1}^{n} |y_k|^2 \geq 0$$

Lemma 2.2. Let A be a matrix with eigenvalues α_k . Define the eigen-values of the Hermitian nonnegative definite matrix A^*A to be $\lambda_k \geq 0$. Then

$$\sum_{k=0}^{n-1} \lambda_k \geq \sum_{k=0}^{n-1} |\alpha_k|^2 \tag{2.8}$$

with equality iff (if and only if) A is normal.

Proof. The trace of a matrix is the sum of the diagonal elements of a matrix. The trace is invariant to unitary operations so that it also is equal to the sum of the eigenvalues of a matrix, i.e.,

$$Tr\left(A^*A\right)=\sum_{k=0}^{n-1}\left(A^*A\right)_{k,k}=\sum_{k=0}^{n-1}\lambda_k. \tag{2.9}$$

From (2.1), A = U RU * and hence

$$Tr\left(A^*A\right)=Tr\left(R^*R\right)=\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\left|r_{j,k}\right|^2.$$

$$=\sum_{k=0}^{n-1}\left|\alpha_k\right|^2+\sum_{k\neq j}\left|r_{j,k}\right|^2.$$

$$=\sum_{k=0}^{n-1}\left|\alpha_k\right|^2 \tag{2.10}$$

Equation (2.10) will hold with equality iff R is diagonal and hence iff A is normal.

Lemma 2.2 is a direct consequence of Shur's theorem ([2], pp. 229-231) and is also proved in [1], p. 106.

2.2 Matrix Norms

To study the asymptotic equivalence of matrices we require a metric on the space of linear space of matrices. A convenient metric for our purposes is a norm of the difference of two matrices. A norm N (A) on the space of n × n matrices satisfies the following properties:

(1). N (A) ≥ 0 with equality if and only if A = 0, is the all zero matrix.
(2). For any two matrices A and B,

$$N(A + B) \le N(A) + N(B). \tag{2.11}$$

(3). For any scalar c and matrix A, N (cA) = |c|N (A).

The triangle inequality in (2.11) will be used often as is the following direct consequence:

$$N(A - B) \ge |N(A) - N(B)|. \tag{2.12}$$

Two norms — the operator or strong norm and the Hilbert-Schmidt or weak norm (also called the Frobenius norm or Euclidean norm when the scaling term is removed) — will be used here ([1], pp. 102–103). Let A be a matrix with eigenvalues α_k and let $\lambda_k \ge 0$ be the eigen-values of the Hermitian nonnegative definite matrix A^*A. The strong norm || A || is defined by

$$\|A\|=\max_x R_{A^*A}\left(x\right)^{1/2}=\max_{z:z^*z=1}\left[z^*A^*Az\right]^{1/2} \tag{2.13}$$

From Lemma 2.1

$$\|A\|^2=\max_k \lambda_k \,\square\, \lambda_M \tag{2.14}$$

The strong norm of A can be bound below by letting e_M be the normalized eigenvector of A corresponding to α_M , the eigenvalue of A having largest absolute value:

$$\|A\|^2 = \max_{z:z^*z=1} z^* A^* A z \geq \left(e_M^* A^*\right)\left(A e_M\right) = \left|\alpha_M\right|^2 . \tag{2.15}$$

If A is itself Hermitian, then its eigenvalues α_k are real and the eigenvalues λ_k of A^*A are simply $\lambda_k = \alpha^2_k$. This follows since if $e^{(k)}$ is an eigenvector of A with eigenvalue α_k , then $A^*Ae^{(k)} = \alpha_k A^*e^{(k)} = \alpha^2_k e^{(k)}$. Thus, in particular, if A is Hermitian then

$$\|A\| = \max_k \left|\alpha_k\right| = \left|\alpha_M\right| \tag{2.16}$$

The weak norm (or Hilbert-Schmidt norm) of an n × n matrix $A = [a_{k,j}]$ is defined by

$$|A| = \left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\left|\alpha_{k,j}\right|^2\right)^{1/2}$$

$$= (\frac{1}{n} Tr\left[A^* A\right])^{1/2} = \left(\frac{1}{n}\sum_{k=0}^{n-1}\lambda_k\right)^{1/2} \tag{2.17}$$

The quantity $\sqrt{n}\,|A|$ is sometimes called the Frobenius norm or Eu-clidean norm. From Lemma 2.2 we have

$$|A|^2 \geq \frac{1}{n}\sum_{k=0}^{n-1}\left|\alpha_k\right|^2 \text{, with equality iff A is normal.} \tag{2.18}$$

The Hilbert-Schmidt norm is the "weaker" of the two norms since

$$\|A\|^2 = \max_x \lambda_k \geq \frac{1}{n}\sum_{k=0}^{n-1}\lambda_k = |A|^2 \tag{2.19}$$

A matrix is said to be bounded if it is bounded in both norms. The weak norm is usually the most useful and easiest to handle of the two, but the strong norm provides a useful bound for the product of two matrices as shown in the next lemma.

Lemma 2.3. Given two n × n matrices $G = \{g_{k,j}\}$ and $H = \{h_{k,j}\}$, then

$$|GH| \leq k\ G\ k\ |H|. \tag{2.20}$$

Proof. Expanding terms yields

$$|GH|^2 = \frac{1}{2}\sum_i\sum_j\left|\sum_k g_{i,k}h_{k,j}\right|^2$$

$$= \frac{1}{n}\sum_i \sum_j \sum_k \sum_m g_{i,k} g^*_{i,m} h_{k,j} h^*_{m,j}$$

$$\lim_{n\to\infty} |A_n| = \lim_{n\to\infty} |B_n|. \quad (2.21)$$

where h_j is the j^{th} column of H. From (2.13),

$$= \frac{h_j^* G^* G h_j}{h_j^* h_j} \le \|G\|^2$$

and therefore

$$|GH|^2 \le \frac{1}{n}\|G\|^2 \sum_j h_j^* h_j = \|G\|^2 |H|^2$$

Lemma 2.3 is the matrix equivalent of (7.3a) of ([1], p. 103). Note that the lemma does not require that G or H be Hermitian.

2.3 Asymptotically Equivalent Sequences of Matrices

We will be considering sequences of n × n matrices that approximate each other as n becomes large. As might be expected, we will use the weak norm of the difference of two matrices as a measure of the "distance" between them. Two sequences of n ×n matrices $\{A_n\}$ and $\{B_n\}$ are said to be asymptotically equivalent if

(1). A_n and B_n are uniformly bounded in strong (and hence in weak) norm:

$$\| A_n \|, \| B_n \| \le M < \infty, n = 1, 2, \ldots \quad (2.22)$$

And

(2) $A_n - B_n = D_n$ goes to zero in weak norm as $n \to \infty$:

$$\lim_{n\to\infty} |A_n - B_n| = \lim_{n\to\infty} |D_n| = 0$$

Asymptotic equivalence of the sequences $\{A_n\}$ and $\{B_n\}$ will be abbreviated $A_n \sim B_n$

We can immediately prove several properties of asymptotic equivalence which are collected in the following theorem.

Theorem 2.1. Let $\{A_n\}$ and $\{B_n\}$ be sequences of matrices with eigenvalues $\{\alpha_n, i\}$ and $\{\beta_n, i\}$, respectively.

(1) If $A_n \sim B_n$, then

$$\lim_{n\to\infty} |A_n| = \lim_{n\to\infty} |B_n|. \quad (2.23)$$

(2) If $A_n \sim B_n$ and $B_n \sim C_n$, then $A_n \sim C_n$.
(3) If $A_n \sim B_n$ and $C_n \sim D_n$, then $A_nC_n \sim B_nD_n$.
(4) If $A_n \sim B_n$ and $\| A^{-1}_n \|, \| B_n^{-1} \| \le K < \infty$, all n, then $A^{-1}_n \sim B_n^{-1}$
(5) If $A_nB_n \sim C_n$ and k A^{-1}_n k≤ K < ∞, then $B_n \sim A^{-1}_n C_n$

(6) If $A_n \sim B_n$, then there are finite constants m and M such that
$m \le \alpha_{n,k}, \beta_{n,k} \le M, \quad n = 1, 2, \ldots \quad k = 0, 1, \ldots, n-1.$ (2.24)

Proof.

(1). Eq. (2.23) follows directly from (2.12).

(2). $|A_n - C_n| = |A_n - B_n + B_n - C_n| \le |A_n - B_n| + |B_n - C_n| \; n\to\infty \; 0$

(3). Applying Lemma 2.3 yields

$$|A_nC_n - B_nD_n| = |A_nC_n - A_nD_n + A_nD_n - B_nD_n| \le k\, A_n\, k\, |C_n - D_n| + k\, D_n\, k\, |A_n - B_n| \xrightarrow[n\to\infty]{} 0$$

(4) $$|A_n^{-1} - B_n^{-1}| = |B_n^{-1}B_nA_n^{-1} - B_n^{-1}A_nA_n^{-1}| \le k\, B_n^{-1}\, k \cdot k\, A_n^{-1}\, k \cdot |B_n - A_n| \xrightarrow[n\to\infty]{} 0$$

(5) $$B_n - A_n^{-1}C_n = A_n^{-1}A_nB_n - A_n^{-1}C_n \le k\, A_n^{-1}\, k\, |A_nB_n - C_n| \xrightarrow[n\to\infty]{} 0$$

(6) If $A_n \sim B_n$ then they are uniformly bounded in strong norm by some finite number M and hence from (2.15), $|\alpha_{n,k}| \le M$ and $|\beta_{n,k}| \le M$ and hence $-M \le \alpha_{n,k}, \beta_{n,k} \le M$. So the result holds for $m = -M$ and it may hold for larger m, e.g., m = 0 if the matrices are all nonnegative definite.

The above results will be useful in several of the later proofs. Asymp-totic equality of matrices will be shown to imply that eigenvalues, prod-ucts, and inverses behave similarly. The following lemma provides a prelude of the type of result obtainable for eigenvalues and will itself serve as the essential part of the more general results to follow. It shows that if the weak norm of the difference of the two matrices is small, then the sums of the eigenvalues of each must be close.

Lemma 2.4. Given two matrices A and B with eigenvalues $\{\alpha_k\}$ and $\{\beta_k\}$, respectively, then

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\alpha_k - \frac{1}{n}\sum_{k=0}^{n-1}\beta_k\right| \le |A - B|$$

Proof: Define the difference matrix $D = A - B = \{d_{k,j}\}$ so that

$$\sum_{k=0}^{n-1}\alpha_k - \frac{1}{n}\sum_{k=0}^{n-1}\beta_k = Tr(A) - Tr(B) = Tr(D).$$

Applying the Cauchy-Schwarz inequality (see, e.g., [22], p. 17) to Tr(D) yields

$$|Tr(D)|^2 = \left|\sum_{k=0}^{n-1} d_{k,k}\right|^2 \le n\sum_{k=0}^{n-1}|d_{k,k}|^2$$

$$\leq n\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\left|d_{k,j}\right|^2 = n^2\left|D\right|^2. \tag{2.25}$$

Taking the square root and dividing by n proves the lemma.
An immediate consequence of the lemma is the following corollary.

Corollary 2.1. Given two sequences of asymptotically equivalent ma-trices $\{A_n\}$ and $\{B_n\}$ with eigenvalues $\{\alpha_{n,k}\}$ and $\{\beta_{n,k}\}$, respectively, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\alpha_{n,k}-\beta_{n,k}\right)=0 \tag{2.26}$$

and hence if either limit exists individually,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{n,k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\beta_{n,k}. \qquad 2.27$$

Proof. Let $D_n = \{d_{k,j}\} = A_n - B_n$. Eq. (2.27) is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}Tr\left(D\right)=0. \qquad 2.28$$

Dividing by n^2, and taking the limit, results in

$$0\leq\left|\frac{1}{n}Tr\left(D\right)\right|^2\leq\left|D_n\right|^2 \xrightarrow[n\to\infty]{} 0 \qquad 2.29$$

from the lemma, which implies (2.28) and hence (2.27).

The previous corollary can be interpreted as saying the sample or arithmetic means of the eigenvalues of two matrices are asymptotically equal if the matrices are asymptotically equivalent. It is easy to see that if the matrices are Hermitian, a similar result holds for the means of the squared eigenvalues. From (2.12) and (2.18),

$$|D_n| \geq ||A_n|| - ||B_n||$$

$$= \left|\sqrt{\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{n,k}^2} - \sqrt{\frac{1}{n}\sum_{k=0}^{n-1}\beta_{n,k}^2}\right|$$

$$\xrightarrow[n\to\infty]{} 0$$

if $|D_n|$ $n\to\infty$ 0, yielding the following corollary.

Corollary 2.2. Given two sequences of asymptotically equivalent Hermitian matrices $\{A_n\}$ and $\{B_n\}$ with eigenvalues $\{\alpha_{n,k}\}$ and $\{\beta_{n,k}\}$, respectively, then

$$lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\alpha_{n,k}^2-\beta_{n,k}^2\right)=0, \qquad 2.30$$

and hence if either limit exists individually,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{n,k}^{2}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\beta_{n,k}^{2}\ . \qquad 2.31$$

Both corollaries relate limiting sample (arithmetic) averages of eigenvalues or moments of an eigenvalue distribution rather than individual eigenvalues. Equations (2.27) and (2.31) are special cases of the following fundamental theorem of asymptotic eigenvalue distribution.

Theorem 2.2. Let $\{A_n\}$ and $\{B_n\}$ be asymptotically equivalent se-quences of matrices with eigenvalues $\{\alpha_{n,k}\}$ and $\{\beta_{n,k}\}$, respectively. Then for any positive integer s the sequences of matrices $\{A^s{}_n\}$ and $\{B_n{}^s\}$ are also asymptotically equivalent,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\alpha_{n,k}^{s}-\beta_{n,k}^{s}\right)=0 \qquad 2.32$$

and hence if either separate limit exists,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\alpha_{n,k}^{s}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\beta_{n,k}^{s}\ . \qquad 2.33$$

Proof. Let $A_n = B_n + D_n$ as in the proof of Corollary 2.1 and consider
$A_n{}^s - B_n{}^s$ □ Δ_n Since the eigenvalues of $A_n{}^s$ are $\alpha_{n,k}{}^s$, (2.32) can be written in terms of Δ_n as

$$\lim_{n\to\infty}\frac{1}{n}Tr\left(\Delta_n\right)=0 \qquad 2.34$$

The matrix Δ_n is a sum of several terms each being a product of D_n's and B_n's, but containing at least one D_n (to see this use the binomial theorem applied to matrices to expand A^s). Repeated application of Lemma 2.3 thus gives

$$\left|\Delta_n\right|\le K\pi\left|D_n\right|\xrightarrow[n\to\infty]{}0 \qquad 2.35$$

where K does not depend on n. Equation (2.35) allows us to apply Corollary 2.1 to the matrices and D^s to obtain (2.34) and hence (2.32).

Theorem 2.2 is the fundamental theorem concerning asymptotic eigenvalue behavior of asymptotically equivalent sequences of matrices. Most of the succeeding results on eigenvalues will be applications or specializations of (2.33).

Since (2.33) holds for any positive integer s we can add sums corresponding to different values of s to each side of (2.33). This observation leads to the following corollary.

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