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APPLICATIONS OF METRIC SPACES

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ABSTRACT

Many of the discussions in several variable calculus is almost identical to the corresponding argument in one variable calculus, especially argument concerning convergence and continuity. We can develop a general notion of distance that covers the distances between numbers, vectors, sequences, functions, sets and much more. Within this theory we can define and prove theorems about convergence and continuity, compactness and boundedness.

KEYWORDS: *Completeness, Continuous Functions, Extension Theorem, Uniform Continuity, Homeomorphism, Separated Sets, Totally Boundedness, Compactness.*

Definition

A function is called a contraction when there is a constant $0 \leq k \leq 1$ such that

$$\forall\, x,y \in X, \quad d(f(x),f(y)) \leq kd(x,y)$$

It follows that f is continuous, because

$$D(x,y) < \delta := \varepsilon/k \Rightarrow d(f(x),f(y)) < \varepsilon$$

Theorem

Let x be a complete metric space, and suppose that f: X→X is a contraction map. Then f has a unique fixed-point x =f(x)

Proof

Consider the iteration $x_{n+1}=f(x_n)$ with $x_0=a$ any point in X. note that,

$$D(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq kd(x_n, x_{n-1})$$

Hence, by induction,

$$D(x_{n+1}, x_n) \leq k^n d(x_1, x_0)$$

The (xn) is a Cauchy sequence since we get, D $(x_n, x_m) \leq d(x_n, x_{n-1}) + \ldots + d(x_{m+1}, x_m)$

$\leq (kn\text{-}1 + \ldots + km)\, d(x_1, x_0) \leq \frac{k^m}{1-k} d(x1\text{-}x0)$

Which converges to 0 as n→∞

Hence xn→x and by continuity of f,

$X = \lim_{n\to\infty} x_{n+1}$

$= \lim_{n\to\infty} f(x_n)$

$= f(\lim_{n\to\infty} x_n)$

$= f(x)$

Moreover, the rate of convergence is given by

$D(xn, x) \leq \frac{k^n}{1-k} d(x, x0)$

Suppose there are two fixed points x=f(x) and y=f(y); then

D(x, y) =d (f(x), f(y))

≤kd(x)

So that d(x, y) =0

Since k<1

Completeness

A metric space (x, d) is said to be complete if every Cauchy sequence in X is convergent.

In other words (x, d) is a complete metric space if, whenever the sequence $\{x_n\}$ in X is such that $d(x_m, x_n) \to 0$ as m, n →∞ there exists an x ∈X with d (xn, x) →0 as n→∞.

Example

Complete metric spaces,

1. The usual metric space Ru and Cu are complete.
2. The discrete metric space Xd is complete
3. The unitary space Cn is a complete metric space

Incomplete metric spaces,

i. The space Qu with the usual metric of absolute value is not complete.
ii. The metric space (X,), where x=] 0, 1] and d is the usual metric on x, is not complete.
iii. The space p [a, b] of all polynomials defined on [a, b] with uniform metric d ∞ is not complete.

Theorem

Let (y, d_y) be a subspace of a metric space (x, d). Then y is complete ⇒ y is closed

Proof

To prove that y is closed, let x be a limit point of y. then, every open sphere centred on x contains points of y. in particular, the open sphere S1/n (x), where n is a positive integer, contains a point x_n of y, other than x. thus $\{x_n\}$ is a sequence in y such that, $x_n \to x$ in X since d(xn,x)< 1/n. let the sequence $\{x_n\}$ is a Cauchy sequence in x and hence in y. but y being complete, x ∈ y.

Hence y is closed.

Dense sets and Separable spaces.

Let (X, d) be a metric space and A ⊂X. the set, A is said to be dense in x if Ā=X.

A metric space (x, d) is said to be separable if it has a countable subset which is dense in X.

Examples

i. The usual metric space Ru is separable since the subset Q ⊂R is countable and dense in R.

ii. The usual metric space Cu is separable since the subset,

S= {a+ib: a, b, ∈ Q}

Is countable and S=c.

iii. The Euclidean space Rn and the unitary space Cn are separable.

iv. The space] p, ≤p≤∞, is separable.

Continuous functions

Let (x, dx) and (y, dy) be metric spaces. A function f: X →Y is continuous at C ∈X if for every ε>0 there δ>0 such that

Dx (x,c)<δ implies that dy (f(x)), fc))<ε

The function is continuous in x if it is continuous at every point of x.

Example

i. In a metric space (X,d), the identity function I: X→X is continuous.

ii. Let f:[0,1] →R be the function given by,

$$F(x) = \begin{cases} 0 \ , \ 0 \leq x \leq 1 \\ 1, \ x = 1 \end{cases}$$

Then, f is continuous in [0, 1] except at x=1.

iii. A function f:R2 →R, where R2is equipped with the Euclidean norm ||I|| and R with the absolute value norm I.I, is continuous at c ∈R2 if ||x-cl|<δ implies that If(x)-f(c) I<ε

Explicitly, if

$X=(x_1, x_2)$, $c= (c_1, c_2)$ and $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))$

This condition reads:

$$\sqrt{(x1-c1)^2+(x2-c2)^2} < \delta$$

Implies that

$$|f(x_1, x_2)-f(c_1, c_2)|<\varepsilon$$

Theorem

A function f: x→y is continuous on x if and only if $f^{-1}(v)$ is open in x for every open set v in y.

Proof

Let f be a continuous function and let v be an open set in y. we shall prove that U=f-1(v) is open in x. let x be any point of U. then f(x) ∈V, which is open. Hence

F(x) ∈B ε (f(x)) ⊂V

And so there exists

.δ>0 $fB_\delta(X) \subset B_\delta$ (f(x)) ⊂V.

In other words, there exists

.δ>0 $B_\delta(x) \subset f^{-1}(v)$ =u

Hence f-1(v) is open.

Conversely, assume that $f^{-1}(v)$ is open in x whenever v is open in y

Let x ∈X be arbitrary and ε> 0 be given, then f(x) ∈y and $B_\varepsilon(f(x))$ is an open set. Then f-1 $B_\varepsilon(f(x))$ is open in x .

I.e., there exists

.δ>0 $x \in B_\delta(x) \subset f^{-1} B_\varepsilon (f(x))$

⇒ there exists

.δ>0$fB_\delta(X) \subset B_\varepsilon (f(x))$

As required

This verifies that f is continuous at x. hence f is continuous.

Theorem

If f is continuous if and only if,

$$(\lim_{n\to\infty} Xn) = \lim_{n\to\infty} f(Xn)$$

Proof

Let f be a continuous function and let (x_n) be a sequence converging to x in the domain. We shall prove that f (X_n) →f(x) in the co-domain as n →∞. Consider the neighbourhood Bε (f(x)) of f(x). Since f is continuous

There exists

.δ > 0fBδ(x) ⊂B_ε (f(x))

But X_n→x means there exists

N>0 n>N ⇒$X_n \in B_\delta(x)$

⇒$f(X_n) \in fB_\delta(x) \subset B_\varepsilon(F(x))$

Conversely, suppose f is not continuous, then there is a point x such that there exists

.ε>0∀δ>0 $fB_\delta(x) \subset B_\varepsilon(F(X))$

In particular there exists

.ε>0 ∀n f $B_{1/n}(x) \subset B_{\varepsilon}(f(x))$

∴ we can find points $X_n \in B_{1/n}(x)$ for which

F(Xn) ∈ Bε(f(x))

i.e., f(Xn) →f(x) while Xn →X

Extension Theorem

If X and Y be any non-empty sets, A ⊂ X and f:A→Y be a function. Then, g: X →Y is called an extension of f to x if f(x) = g(x), ∀ x ∈ A and f is called the restriction of g to A, denoted by $g/_A$ or g A.

Uniform Continuity

Let (X,dx) and (Y,dy) be metric spaces. A function f:X→Y is uniformly continuous on X if for every ε >0 there exists δ> 0 such that

Dx (x,y) <δ⇒ dy (f(x), f(y)) <ε

Example

Let f: R→R by f (x) =X2. It is easy to verify that f is continuous. We shall prove that f is not uniformly continuous. Then there exists an ε > 0 for which no δ works. Take ε = 1. Let δ > 0 be given. Let

$$X1=\frac{\delta}{2} + \frac{1}{\delta},\ X2 = \frac{1}{\delta}$$

Then

$$|X1 - X2| = \frac{\delta}{2}<\delta$$

But,

$$|f(x1) - f(x2)| = |(\frac{\delta}{2}+\frac{1}{\delta})\ 2\ -\frac{1}{\delta 2}|$$

$= |\frac{\delta 2}{4} + 1|$

$= \frac{\delta 2}{4} + 1 > 1$

Thus, whatever δ may be, there exists x1, x2∈ R such that |x1 – x2 | < δ, but |f (x1) –f (x2)|>1 .

Homeomorphism

Let (X, d) and (Y, P) be teo metric spaces. A function f: X →Y is said to be a homeomorphism if

i. f is bijective
ii. f is continuous
iii. f¯1 is continuous

If a homeomorphism from X to Y exists, we say that the spaces X and Y are homeomorphic.

Examples

i. The metric space [0,1] and [0,2] with the usual metric are homeomorphic. Indeed, ifg f(x) = 2x, then f is a homeomorphism of [0,1] onto [0,2].
ii. The usual metric space Ruand the discrete metric space Rdare not homeomorphic.

Separated Sets

Let (X,d) be a metric space and A,B ⊂ X . The sets A and B are sais to be separated if A ∩B = ∅ and Ā∩B = ∅.

Examples

i. In the usual metric space Ru, the sets A =]0,1[and B =]1,2[are separated.

ii. In general, any two disjoint sets in Rd are separated.

Connectedness A = B U C and each subset can be covered exclusively by an open set.

i.e., B ⊆ U, C ∩U = ∅

C ⊆ V, B ∩V = ∅

A set is called connected otherwise.

Examples

i. Any subset of natural numbers is disconnected except the single points {n} and the empty set.

ii. The set of rational numbers Q is disconnected.

i.e., Q⊆ $(-\infty,\sqrt{2})$ U $(\sqrt{2},\infty)$

Totally Boundedness

A set B is totally bounded with when ∀ε> 0 there exists a1, ……, aN

B $\subseteq \cup_{i=1}^{N}(ai)$

i.e., a set is totally bounded when it can be covered by a finite number of ε - balls, however small their radii ε.

Examples

The set [0,1] is totally bounded because it can be covered by the balls Bε (nε) for n= 0,… … . , N where N $>\frac{1}{\varepsilon}$.

Theorem

Let (X,d) be a metric space. If X is totally bounded, then X is bounded.

Proof

Since X is totally bounded, for each ε> 0, it has a finite ε – net, in particular, it has a finite 1 – net A.

Let A = {a1, a2, … … … … , aN

Then

$$X = \cup_{i=1}^{N} S1(ai)$$

Since finite union of bounded sets is bounded, it follows that X is bounded.

Compactness

A set K is said to be compact if

$$K \subseteq \cup_i B\varepsilon i(ai) \implies \exists = i\ 1, \dots, iN$$

i.e., K $\subseteq \cup_{K=1}^{N} B\varepsilon ik(aik)$

Examples

The set [0,1] is not compact. For example, the cover of balls B1$\frac{1}{n}$ (0) for n= 2, … … has no finite subcover. Similarly, the sets R and not compact. On the other hand, we will soon see that the sets [a, b] are compact in R.

Theorem

Let K be a compact metric space and Y a metric space. If f: K →Y is a continuous function, then f(K) is a compact subset of Y.

Proof

Let Ai be an open cover for f(K) . We shall prove that a finite subcollection of them still covers f (K).

From

$$F(K) \subseteq \cup_i Ai$$

We can deduce

$$K \subseteq f^{-1} \cup_i Ai = \cup_i f^{-1} Ai$$

But f^{-1} Ai are open sets since f is continuous

$$\therefore \text{ The right hand side is an open cover of } K \text{ , which is compact .}$$

∴ The finite number of these open sets will do to cover K.

$$K \subseteq \cup_{i=1}^{N} f^{-1} Ai$$

It follows that

$$F(K) \subseteq \cup_{i=1}^{N} Ai$$

i.e., a finite number of the original open sets Ai will cover the sets f (K), which is therefore compact.

Bolzano - Weierstrass Compact Set

A metric space (X,d) is said to have Bolzano - Weierstrass property if every infinite subset of X has a limit point in S .

Example

Consider the metric space (X,d) , where X =]0,1[and d is the usual metric , S = X and the infinite subset A = { 1, $\frac{1}{2}$, $\frac{1}{3}$, … … … } of S. Here 0 is the only limit point of the set A and this is not in S.

The Minknowski Inequality

The set Rn with the ep norm defined for x = (X1, X2, ………, Xn) and

$1 \leq p \leq \infty$ by

$|| x ||p = (| x1|p + ………. + ||xn |p |)^{\frac{1}{p}}$

And for p = ∞ by

$||x||_\infty = \max(|x1|, |x2| p, \ldots\ldots\ldots, |xn|p)$

Is an n – dimensional normed vector space for every 1 ≤p ≤∞. The Euclidean case p= 2 is distinguished by the fact that the norm ||. ||2is derived from an inner product on Rn.

$|| x ||2 = \sqrt{\langle x, y \rangle}$

$\langle x,y \rangle = \sum_{i=1}^{n} xiyi$

The triangle inequality for the ep – norm is called Minkowski's inequality.

Theorem

If (x1, x2, ……., xn), (y1, y2, …….,yn) ∈Rn , then $|\sum_{i=1}^{n} XiYi| \leq (\sum_{i=1}^{n} Xi\ 2)^{\frac{1}{2}} (\sum_{i=1}^{n} Yi2\ )^{\frac{1}{2}}$

Proof

Since |∑ xiyi|≤∑|xi| |yi|, it is sufficient to prove the inequality for xi, yi ≥ 0. Furthermore, the inequality is obvious if either x =0 or y = 0, So we assume at least one xi and one y1 is non zero.

For every $\alpha, \beta \in$R, we have

$$0 \leq \sum_{i=1}^{n} (\alpha xi - \beta yi)^2$$

Expanding the square on the right-hand side and re arranging the terms, we get that

$$2\alpha\beta = \sum_{i=1}^{n} xiyi \leq \alpha^2 \sum_{i=1}^{n} xi^2 + \beta^2 \sum_{i=1}^{n} yi^2$$

Then division of the resulting inequality by $2\alpha\beta$ proves the theorem

Then MInkowski inequality for p = 2 is immediate consequence of Cauchy Schwartz inequality.

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