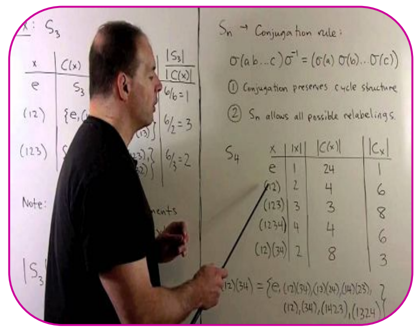
REVIEW OF RESEARCH
IMPACT FACTOR : 5.2331(UIF) UGC APPROVED JOURNAL NO. 48514 ISSN: 2249-894X
VOLUME - 7 | ISSUE - 9 | JUNE - 2018

SIMPLEST WAY FOR FINDING CLASS EQUATION OF A FINITE GROUPS FROM THE NORMALIZER

Upendra Kumar
Assistant Professor , Department of Mathematics ,
Govt. Post Graduate College Obra,Sonebhadra , Uttar Pradesh.

ABSTRACT:

The Concept of class equation of a finite group plays a central role in the group theory. For finding the class equation of a finite group means we have to find out total number of conjugacy class of a finite group and then total number of elements in each conjugacy classes. If the group is abelian then the class equation is trivial. However, if the group is not abelian then the number of conjugacy classes is less than the order of the group. In this paper we count the conjugacy classes for finite non abelian groups and total number of elements in each conjugacy classes using normalize of an element of a group.



KEYWORDS : *class equation , conjugacy classes , central role.*

INTRODUCTION

1.1 Group: A non empty set G equipped with an operation '*' is said to be group if it satisfies the following postulates.

1. **Closure Property:** *i.e.* $\forall \; a,b \in G \Rightarrow a*b \in G.$

2. **Associativity** : *i.e.* $a*(b*c) = (a*b)*c \quad \forall a,b,c \in G.$

3. **Existence of identity element:** If $a \in G$ the there exists an element $e \in G$ such that

$$e*a = a = a*e \quad \forall a \in G.$$

4. **Existence of inverse element:** If $a \in G$ the there exists an element $a^{-1} \in G$ such that

$$a^{-1}*a = e = a*a^{-1} \quad \forall a \in G.$$

1.2 Finite and Infinte Group: A group G is said to be finite group if the number of distinct elements in group is finite otherwise G is an infinite group.

1.3 Order of Group: The number of elements in a finite group is called the order of the group and is denoted by $O(G)$. An infinite group is said to be of infinite order.

1.4 Subgroup: A non empty subset H of a group G is called a subgroup of G if H itself is a group w.r.t. the same composition as defined in G.

1.5 Cosets: Let H be a subgroup of a group G and $a \in G$. The set

$aH = \{ ah \ : \ h \in H \}$ is called a left coset of H in G.
$Ha = \{ ha \ : \ h \in H \}$ is called a right coset of H in G.

1.6 Index of a subgroup in a group: The index of a subgroup H of a group G is defined as the number of distinct right (or left) cosets of H in G. It is denoted by $I_G(H)$ *or* $[G:H]$.

1.7 Normalizer of an element a of a group: The normalize *N(a)* of an element *a* of a group G is the set of all elements *x* in G that commute with *a*. *i.e.*

$$N(a) = \{x \in G \ : \ xa = ax\}$$

1.8 Centre of a group: The centre *Z(G)* of a group G is the set of all elements z in G that commute with every element *x* in G. *i.e.*

$$C(Z) = \{z \in G \ : \ zx = xz \quad \forall x \in G\} \qquad .$$

1.9 Conjugate element in a group: If *a* and *b* be any two elements of a group G then *a* is said to be conjugate to *b* if and only if

$$a = x^{-1}bx \ \ for\ some\ x \in G.$$

If *a* is conjugate to *b* then it is denoted by $a \sim b$. The relation '$\sim$' is known as relation of conjugacy.

1.10 Conjugate class or Equivalence class: Since the relation of conjugacy in a group G is an equivalence relation. Therefore it will partition the group G into disjoint classes called conjugate classes or equivalence classes.

The collection of all elements conjugate to an element $a \in G$ is called conjugate class of *a*. It is denoted by C_a *or* $C(a)$ *or* $Cl(a)$ *or* $[a]$. Thus

$$\begin{aligned} C(a) &= \{x \in G : x \sim a\} \\ &= \{x \in G : x = y^{-1}ay \text{ for some } y \in G\} \\ &= \{y^{-1}ay : y \in G\} \\ &= \{x^{-1}ax : x \in G\} \end{aligned}$$

1.11 Self conjugate elements: An element a of a group G is said to be self conjugate element if a is the only member of the class $C(a)$ of elements conjugate to a. *i.e.*

$$C(a) = \{a\}.$$

Thus a is self conjugate element *iff*

$$\begin{aligned} a &= x^{-1}ax \quad \forall x \in G \\ \Rightarrow xa &= ax \quad \forall x \in G. \end{aligned}$$

1.12 Theorem: The relation of conjugacy is an equivalence relation.

Proof: Let G be a group and '$\sim$' is the relation of conjugacy in a group. Then we shall prove that '$\sim$' is an equivalence relation. For,

1. **Reflexivity:** Let $a \in G$ then we have

$$a = e^{-1}ae \quad \text{where } e \in G \text{ is the identity element of } G.$$
$$\Rightarrow a \sim a.$$

$$\text{Thus } a \sim a \quad \forall\, a \in G.$$

Therefore the relation is reflexive.

2. **Symmetry:**

$$\text{Let } a \sim b \text{ then}$$

$$\begin{aligned} & a = x^{-1}bx \qquad \text{for some } x \in G \\ \Rightarrow\ & xa = xx^{-1}bx \\ \Rightarrow\ & xax^{-1} = xx^{-1}bxx^{-1} \\ \Rightarrow\ & xax^{-1} = ebe \\ \Rightarrow\ & xax^{-1} = b \\ \Rightarrow\ & b = xax^{-1} \\ \Rightarrow\ & b = \left(x^{-1}\right)^{-1}ax^{-1} \quad \text{where } x^{-1} \in G \end{aligned}$$

$$Thus\ \ a \sim b \Rightarrow b \sim a \qquad \forall\ a,b \in G.$$

Therefore the relation is symmetric.

3. Transitivity:

$$Let\ a \sim b\ and\ b \sim c\ then$$

$$a = x^{-1}bx \qquad for\ some\ x \in G \quad ...(i)$$

$$b = y^{-1}cy \qquad for\ some\ y \in G \quad ...(ii)$$

$$from\ (i)\ we\ have$$

$$a = x^{-1}bx$$

$$\Rightarrow a = x^{-1}(y^{-1}cy)x \qquad <\ from\ (ii)>$$

$$\Rightarrow a = x^{-1}y^{-1}cyx$$

$$\Rightarrow a = (yx)^{-1}c(yx) \qquad where\ xy \in G$$

$$\Rightarrow a \sim c$$

$$Thus \qquad a \sim b\ and\ b \sim c \Rightarrow a \sim c$$

Therefore the relation is transitive.

Since the relation of conjugacy is reflexive, symmetric and transitive. Therefore the relation of conjugacy is an equivalence relation.

1.13 Theorem: If Z be the centre of a group G and $a \in G$ then $a \in Z$ *iff* $N(a) = G$. Also if G is finite then $a \in Z$ *iff* $O[N(a)] = O(G)$.

Proof: Let Z be the centre of a group and $N(a)$ be the normalize of an element $a \in G$ then

$$Z = \{z \in G :\ zx = xz \quad \forall\ x \in G\}$$

$$and \qquad N(a) = \{x \in G : ax = xa\}$$

$$Let\ a \in Z\ then\ \ ax = xa \qquad \forall\ x \in G$$

$$\Leftrightarrow x \in N(a) \qquad \forall\ x \in G$$

$$\Leftrightarrow N(a) = G \qquad <\ \because\ N(a) \subseteq G\ and\ each\ element\ of\ G\ is\ in\ G\ >$$

If G is a finite group, then

$$N(a) = G$$
$$\Rightarrow O[N(a)] = O(G)$$

CLASS EQUATION OF A FINITE FROUP

Let G be a finite group. Let $N(a)$ be the normalize of an element $a \in G$ and $C(a)$ be the conjugate class of a in G then

$$N(a) = \{x \in G : ax = xa\}$$

$$and \qquad C(a) = \{x^{-1}ax : x \in G\}$$

Let $\sum$ *denote the set of all right cosets of* $N(a)$ *in* G

i.e., $\sum = \{N(a)x : x \in G\}$

Define a mapping $f : N(a) \to \sum$ *such that*

$$f(x^{-1}ax) = N(a)x \quad , x \in G \qquad ...(1)$$

Then

1. f is well defined mapping:

Let $x^{-1}ax,\ y^{-1}ay \in C(a)$ *such that*

$$x^{-1}ax = y^{-1}ay$$
$$\Rightarrow xx^{-1}ax = xy^{-1}ay$$
$$\Rightarrow ax = xy^{-1}ay$$
$$\Rightarrow axy^{-1} = xy^{-1}ayy^{-1}$$
$$\Rightarrow axy^{-1} = xy^{-1}a$$
$$\Rightarrow xy^{-1} \in N(a) \qquad <\because \text{ if } H < G \text{ then } Ha = aH \Leftrightarrow ab^{-1} \in H$$
$$\Rightarrow N(a)x = N(a)y$$
$$\Rightarrow f(x^{-1}ax) = f(y^{-1}ay)$$

Thus $x^{-1}ax = y^{-1}ay \Rightarrow f(x^{-1}ax) = f(y^{-1}ay)$

Therefore f *is well defined mapping.*

2. **f is one one mapping:**

$Let\ f\left(x^{-1}ax\right), f\left(y^{-1}ay\right) \in \sum$ *such that*

$$f\left(x^{-1}ax\right) = f\left(y^{-1}ay\right)$$
$$\Rightarrow N(a)x = N(a)y$$
$$\Rightarrow xy^{-1} \in N(a) \qquad <\because \text{ if } H < G \text{ then } Ha = aH \Leftrightarrow ab^{-1} \in H >$$
$$\Rightarrow a\left(xy^{-1}\right) = \left(xy^{-1}\right)a$$
$$\Rightarrow axy^{-1} = xy^{-1}a$$
$$\Rightarrow x^{-1}axy^{-1} = x^{-1}xy^{-1}a$$
$$\Rightarrow x^{-1}axy^{-1} = y^{-1}a$$
$$\Rightarrow x^{-1}axy^{-1}y = y^{-1}a\ y$$
$$\Rightarrow x^{-1}ax = y^{-1}a\ y$$

Thus $f\left(x^{-1}ax\right) = f\left(y^{-1}ay\right) \Rightarrow x^{-1}ax = y^{-1}ay$

Therefore f *is one one mapping.*

3. **f is onto mapping:**

Let $N(a)x \in \sum$ *then there esxits* $x^{-1}ax \in C(a)$ *such that*

$$f\left(x^{-1}ax\right) = N(a)$$

Therfore f *is onto.*

Thus f *is one one onto mapping from* $C(a)$ *to* $\sum$ *and* G *is finite. Hence*

$$O\left[C(a)\right] = O\left(\sum\right)$$
$$\Rightarrow O\left[C(a)\right] = I_G\left[N(a)\right]$$
$$\Rightarrow O\left[C(a)\right] = \frac{O(G)}{O\left[N(a)\right]} \qquad \ldots(2)$$

Since the relation of conjugacy is an equivalence relation on G. Therefore it partitions G into disjoint equivalence classes. But G is finite group. Therefore the number of distinct conjugate classes of G will be finite, say, equal to k. Let $C(a_1), C(a_2), C(a_3), \ldots, C(a_k)$ are the k distinct conjugate classes of G then

$$G = C(a_1) \cup C(a_2) \cup C(a_3) \cup \ \ldots \cup C(a_k)$$

$$\Rightarrow O(G) = O[C(a_1)] + O[C(a_2)] + O[C(a_3)] + \ \ldots + O[C(a_k)]$$

$$\Rightarrow O(G) = \sum_{a \in G} O[C(a)]$$

$$\Rightarrow O(G) = \sum_{a \in G} \frac{O(G)}{O[N(a)]} \qquad < \text{from equation}(2) >$$

where the sum runs over one element a in each conjugate class

$$\Rightarrow O(G) = \sum_{a \in Z} \frac{O(G)}{O[N(a)]} + \sum_{a \notin Z} \frac{O(G)}{O[N(a)]} \qquad \ldots(3)$$

Now $a \in Z \Leftrightarrow O[N(a)] = O(G)$

$$\Leftrightarrow \frac{O(G)}{O[N(a)]} = 1$$

$$\Leftrightarrow O[C(a)] = 1$$

$\Leftrightarrow$ *The conjugate class of a in G containing only one element. Thus the number of conjugate classes each having only one element is equal to* $O(Z)$.

Hence

$$O(Z) = \sum_{a \in Z} \frac{O(G)}{O[N(a)]} \qquad \ldots(4)$$

From equation (3) *and* (4) *we have*

$$O(G) = O(Z) + \sum_{a \notin Z} \frac{O(G)}{O[N(a)]}$$

where the summation runs over one lelment a in each conjugate class containing more than two element.

Example: Find class equation of non ableian Dihedral Group D_4.

Solution:

$$\begin{aligned}
&Since && O\left(D_4\right) = 2 \times 4 = 8 = 2^3 \\
&Therefore && O\left(Z\right) = 2 \\
&and && O\left[N\left(a\right)\right] = 4 \quad where\ a \in D_4
\end{aligned}$$

$Hence\ class\ eqation\ of\ D_4\ \ is:$

$$O\left(D_4\right) = 1 + 1 + \sum_{a \notin Z} \frac{8}{4}$$
$$\Rightarrow O\left(D_4\right) = 1 + 1 + \sum_{a \notin Z} 2$$
$$\Rightarrow O\left(D_4\right) = 1 + 1 + \left(2 + 2 + 2\right).$$

SOME OTHER EXAMPLES:

1. $o(G) = 1$ *then class equation of* G : $o(G) = 1.$
2. $o(G) = 2$ *then class equation of* G : $o(G) = 1 + 1.$
3. $o(G) = 3$ *then class equation of* G : $o(G) = 1 + 1 + 1.$
4. $o(G) = 4$ *then class equation of* G : $o(G) = 1 + 1 + 1 + 1.$
5. $o(G) = 5$ *then class equation of* G : $o(G) = 1 + 1 + 1 + 1 + 1.$
6. $o(G) = 6 = o(S_3)$ *then class equation of* G : $o(G) = 1 + 2 + 3.$
7. $o(G) = 7$ *then class equation of* G : $o(G) = 1 + 1 + 1 + 1 + 1 + 1 + 1.$
8. $o(G) = 8 = o(D_4)$ *then class equation of* G : $o(G) = 1 + 1 + 2 + 2 + 2.$

REFERENCES

1. Abdollahi, A., Jafarian, S.M.A. and Mohammadi,H.A. (2007). Groups with specific number of centralizers. Houston Journal of Mathematics, 33(1), 43-57.
2. Herstein, I. N. (1964). Topics in Algebra. Massachusetts USA: Blaisdell Publishing Company.
3. Houshang, B. and Hamid, M. (2009). A note on p-groups of order ≤ p4. Proc. Indian Acad. Sci. (Math. Sci.), 119(2), 137-143.
4. Jelten, N.B. and Momoh, S.U. (2014). Minimum and maximum number of irreducible representations of prime degree of non abelian group using the centre. Journal of Natural Sciences Research. International Institute of Science and Technical Eduction, 4(10), 63 – 69.
5. Louis, S. (1975). Introduction to abstract algebra. New York: McGraw – Hill Inc. Mann, A. (2011). Conjugacy classes of finite p – groups. 7p. Retrieved on 12/2/2014. Mark, R. (2011).

Notes on group theory. https://www2,bc.edu/-/development-group- Theoryhtml, 4-9. Retrieved on 9/9/ 2012.

6. Sarah, M. B. and Gary, J. (1991). Counting centralizers in finite groups. Rose –Hulman Institute of Technology Terre Haute, 15.