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SUMUDU TRANSFORM FOR SOLVING SOME CLASSES OF FRACTIONAL DIFFERENTIAL EQUATIONS

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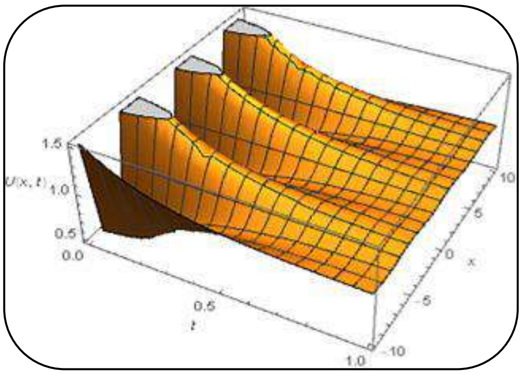
ABSTRACT

In this paper, some properties of fractional Sumudu transform for solving fractional differential equations (FDEs) are presented. Approximate solutions for some classes of FDEs have been studied using the Sumudu transform method. The purpose of this work is to show the advantages of using the Sumudu transform method and the expansion of coefficients of binomial series to solve fractional differential equations.



KEYWORDS: *Sumudu Transform, Laplace Transform, Fractional Differential Equations, Binomial Series, Models, Class.*

INTRODUCTION

Differential equation are useful tools in mathematical models of real-life problems and in applied mathematics. Differential equations have played a very important role in various applications of mathematics and with the development of computers. Thus, the investigation and analysis of differential equations led to many mathematical problems. Therefore, authors have proposed many transformation techniques to solve different types of Differential equations such as: Laplace, Fourier, Mellin, Hankel transformations and Sumudu transforms which are little known and still not widely used to solve differential equations. The single Sumudu transform (or Sumudu transform) was originally proposed by for exponential order functions to solve differential equations and control engineering problems and proved many interesting properties in the T-domain and U-domain, as well as properties and applications of Sumudu. Transform the ODEs described by it. Among integral transforms, the Sumudu transform has unique preserving properties and can therefore be used to solve problems without resorting to them in the frequency domain, and this is one of the many power points of this new transform, especially with physical dimension problems.

Nonlinear PDEs appearing in many branches of physics, engineering, and applied mathematics cannot be successfully described, thus they are described by models of fractional calculus, derived formulas for the single Sumudu transform of partial derivatives and also applied them to solving initial value problems (IVP), studied applications of the Sumudu transform in solving PDEs. The Sumudu transform of partial derivatives is derived by [2] and its applicability is demonstrated using three different PDEs. [3] Expanded Sumudu transforms between functions of two variables. Using this extended definition, a function of two variables F(x, y) is transformed into a function such as F(u, v) with emphasis on the solutions of the PDE. [4] presented an analytical investigation of the Sumudu transform and applications to integral product equations. Laplace transform is the dual of Sumudu transform for solving mathematical problems.

However, the Sumudu transform can be used to solve mathematical problems without resorting to a new frequency domain. In the last two decades the subject of fractional calculus has been extensively investigated and has gained considerable importance and popularity. [5] and [6] have studied approximate solutions of FDEs of Lane-Emden type by collocation method and least square method respectively. [7] extended the theory and applications of the Sumudu transform and used it to solve FDEs by direct integration methods. [8] studied and proved the Sumudu transform properties and the Laplace-Sumudu transform duality and complex inversion formulas while [9] developed analytical methods for solving FPDEs and developed an extended Sumudu transform iteration method for solving time and space FPDEs. Also, their system. They demonstrated the usefulness of the method by finding exact solutions for a large number of FPDEs. [10] used Sumudu transform technique to solve difficult problems for DEs with Caputo fractional derivative and solution of fractional diffusion-wave equation while [11] used Sumudu transform and variational iteration method (VIM) to approximate. Entropy, wavelets etc. [18] obtained approximate solutions of some homogeneous Fr-PDEs by applying the Laplace of fractional derivatives and expanding the coefficients of the binomial series to the solutions of the corresponding FDEs. Finally, many researchers studied the fractional Sumudu transform in [12, 13, 14, 15, 16, 17]. In this paper, we derive a standard formula for finding approximate solutions of some classes of homogenous FPDEs using Sumudu fractional derivatives and expansion of coefficients in binomial series. We solve the same equations as solved in [18] using Laplace. transform but we use Sumudu transform instead and we get the same results.

Preliminary

In this section, we present the preliminary concepts and some definitions for this study.

Definition 2.1. Fractional Derivatives of Casual Functions [18]:

A fractional derivative of a periodic signal is periodic if it is defined over the entire real line. The fractional derivative of a causally periodic signal is never causally periodic. The fractional derivative of a logarithmic function can be approximated using the Caputo definition. These concepts have practical applications, such as in the design of analog signal processing circuits.

The fractional derivative of a casual function f(t) is defined by [19] as follows:

$$(1) \qquad \frac{d^\alpha}{dt^\alpha} f(t) = \begin{cases} f^{(n)}(t), & \text{if } \alpha = n \in N, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, & \text{if } n-1 < \alpha < n, \end{cases}$$

where the Euler gamma function Γ (.) is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ (R(z) > 0).$$

Definition 2.2 Sumudu Transform Method:

Sumudu Transform is introduced by Watugula in [1]. The Sumudu transform can be used to solve problems without resorting to a new frequency domain. Due to its simple formulation and consequent special and useful properties, the Sumudu transform has already shown a lot of promise. It has been demonstrated here and elsewhere that engineering mathematics can help solve complex problems in the applied sciences. However, despite the potential introduced by this new operator, only a few theoretical investigations have appeared in the literature over a period of fifteen years. Most transform theory books available, if not all, do not refer to the Sumudu transform. Even relatively recent well-known comprehensive handbooks, such as those of Debnath and Poulrikas, do not find mention of the Sumudu transformation.

Sumudu transform of a function [20-21]. Let f (t) be a real function defined on the domain (0, ∞) Sumudu transform of f (t) is defined as:

$$G(u) = S[f(t)] = \int_0^\infty e^{-t} f(ut)\, dt, u \in C$$

provided the integral exists for some u.

Definition 2.3 Mitag-Leffler function:

The Mittag-Leffler function is defined by [23].

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+\beta)}, \quad (z, \alpha, \beta \in C, R(\alpha > 0).$$

Definition 2.4 (The Simplest Right Function). The simplest write function [18] is defined as:

$$\phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\beta)} \cdot \frac{z^k}{k!}, (z, \alpha, \beta \in C).$$

Definition 2.5 (The Riemann-Liouville Fractional Derivatives).

The Riemann- Liouville fractional derivatives $D_a^\alpha + \gamma$and $D_a^\alpha - \gamma$ of order a Є C(R(a) ≥ 0) are defined by [24-25] as follows:

$$(D_{a+}^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}}; \qquad n = [R(\alpha)] + 1; x > a$$

And

$$(D_{b-}^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)} (-\frac{d}{dx})^n \int_a^x \frac{y(t)dt}{(t-x)^{\alpha-n+1}}; \qquad n = [R(\alpha)] + 1; x > b$$

respectively, where [R(α)] means the integral part of R(α).

Definition 2.6 (Pochhammer symbol). The Pochhammer symbol (or shifted factorial, since $(1)_n$ = n! of n Є N0 ={0,1,2,...}) [26-27] is given by:

$$(\lambda)_n = \begin{cases} 1, & (n = 0), \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & (n \in N_0/\{0\}). \end{cases}$$

Definition 2.7 (binomial coefficient). The binomial coefficients are defined by [18] as follows:

$$\binom{\lambda}{n} = \frac{\lambda!}{\lambda!(\lambda-n)!} = \frac{\lambda(\lambda-1)(\lambda-n+1)}{n!},$$

where λ and n are integers.

Observe that 0!=1, then,

$\binom{\lambda}{0} = \binom{\lambda}{\lambda}$ =1

And

$$(1-z)^{-\lambda} = \sum_{r=0}^{\infty} \frac{(\lambda)r}{r!} z^r = \sum_{r=0}^{\infty} \binom{\lambda+r-1}{r} z^r.$$

Fractional Sumudu transform

Using the definition of the fractional Sumudu transform, we can easily derive the following operational formulas:

1. $S_\alpha\{f(at)\} = G_\alpha(au)$,
2. $S_\alpha\{f(t-b)\} = E_\alpha(-b^\alpha) = G_\alpha(u)$,
3. $S_\alpha\{E_\alpha(-c^\alpha t^\alpha)\} = \frac{1}{(1+cu)^\alpha} G_\alpha(\frac{u}{1+cu})$,
4. $S_\alpha\{\int_0^t f(t)(dt)^\alpha\} = u^\alpha \Gamma(1+\alpha) G_\alpha(u)$,
5. $S_\alpha\{f^\alpha(t)\} = \frac{G_\alpha(u) - \Gamma(1+\alpha) f(0)}{u^\alpha}$,
6. $S_\alpha^2\{f(at, bx)\} = G_\alpha(au) H_\alpha(bv)$,
7. $S_\alpha^2\{f(at)g(bx)\} = G_\alpha^2(au, bv)$,
8. $S_\alpha^2\{f(t-a, x-b)\} = E_\alpha(-(A+B)^\alpha) G_\alpha^2(au, bv)$,
9. $S_\alpha^2\{\partial_t^\alpha f(t,x)\} = \frac{G_\alpha^2(u,v) - \Gamma(1+\alpha) f(0,x)}{u^\alpha}$,
10. If one defines the rotation expression of the order of two functions f(t) and g(t)

$$(f(x) * g(x))_\alpha = \int_0^x f(x-v) g(v) (dv)^\alpha,$$

then $S_\alpha\{(f(t) * g(t))_\alpha\} = u^\alpha G_\alpha(u) H_\alpha(u)$, where $G_\alpha(u) = S_\alpha\{f(t)\}$ and $H_\alpha(u) = S_\alpha\{g(t)\}$.

and for the detailed proof of the above properties.

3. Method of solution of fractional differential equations

Throughout this section, consider y(t) such that for some value of the parameter u, the Sumudu transform G(u) = S(y(t)) converges.

Theorem 3.1. Let o < α< 1 and b ∈ R. Then, the FDE
(2) y^α (t) — by(t) = 0, t ≥ 0.
with the initial condition y(0) = c_0 the solution is

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(bu^\alpha)^k}{\Gamma(\alpha k + 1)}.$$

Proof. Applying Sumudu transform for Equation (2)

$$\frac{G(u)-G(0)}{u^{\alpha}} - bG(u) = 0.$$

Then

$$G(u) = \frac{c_0}{1-bu^{\alpha}}.$$

Since

$$\frac{1}{1-bu^{\alpha}} = \sum_{k=0}^{\infty}(bu^{\alpha})^k.$$

So,

(3) $$G(u) = c_0\sum_{k=0}^{\infty}(bu^{\alpha})^k.$$

Taking inverse Sumudu transform for Equation (3)

$$y(t) = c_0\sum_{k=0}^{\infty}\frac{(bu^{\alpha})^k}{\Gamma(\alpha k+1)}.$$

Theorem 3.2. Let 1 < a < 2 and a, b ∈ R. Then, the following FDE

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0.$$

with the initial conditions $y(0) = c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+1)r!} \\ &+ c_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+2)r!} \\ &+ ac_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{k+\alpha-1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+\alpha)r!}. \end{aligned}$$

Proof. Applying Sumudu transform for Equation (4)

$$\frac{G(u)-G(0)}{u^{-\alpha}} - \frac{G'(0)}{u^{\alpha-1}} + \frac{a(G(u)-G(0))}{u} + bG(u) = 0.$$

That is,

$$G(u)(u^{-\alpha} + au^{-1} + b) = (c_0u^{-\alpha} + c_1u^{1-\alpha} + ac_0u^{-1}),$$
$$G(u) = \frac{c_0u^{-\alpha} + c_1u^{1-\alpha} + ac_0u^{-1}}{u^{-\alpha} + au^{-1} + b}.$$

Since

$$\begin{aligned}
\frac{1}{u^{-\alpha}+au^{-1}+b} &= \frac{u}{u^{1-\alpha}+a+bu}, \\
&= \frac{u}{(u^{1-\alpha}+a)(1+\frac{bu}{u^{1-\alpha}+a})}, \\
&= \frac{u}{u^{1-\alpha}+a}\sum_{k=0}^{\infty}(\frac{-bu}{u^{1-\alpha}}+a)^k, \\
&= \sum_{k=0}^{\infty}\frac{(-b)^k u^{\alpha k+\alpha}}{(1+au^{\alpha-1})^{k+1}}, \\
&= \sum_{k=0}^{\infty}(-b)^k u^{\alpha k+\alpha}\sum_{r=o}^{\infty}(-au^{\alpha-1})^r\binom{k+r}{r}, \\
&= \sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha}.
\end{aligned}$$

Then,

$$\begin{aligned}
G(u) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k} \\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+1} \\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}u^{(\alpha-1)r+\alpha k+\alpha-1}. \qquad (5)
\end{aligned}$$

After taking the inverse Sumudu transform for equation (5), we get the solution

$$\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+1)r!} \\
&+ c_1\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+2)r!} \\
&+ ac_0\sum_{k=0}^{\infty}\frac{(-b)^k t^{\alpha k+\alpha-1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)(-at^{\alpha-1})^r}{\Gamma((\alpha-1)r+\alpha k+\alpha)r!}
\end{aligned}$$

Example 3.1. If we let $\alpha = \frac{3}{2}$, $a = -1$ and $b = -2$ in Theorem 3.2, then, the following FDE

$$y^{(\frac{3}{2})}(t) - y'(t) - 2y(t) = 0, \quad t \geq 0. \qquad (6)$$

has a solution

$$\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}\frac{(2)^k t^{\frac{3}{2}k}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+1)r!} \\
&+ c_1\sum_{k=0}^{\infty}\frac{(2)^k t^{\frac{3}{2}k+1}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+2)r!} \\
&- c_0\sum_{k=0}^{\infty}\frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!}\sum_{r=0}^{\infty}\frac{\Gamma(k+r+1)t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+\frac{3}{2})r!}. \qquad (7)
\end{aligned}$$

[18] solved equation (6) using the Laplace transform with the following solution

$$
\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+1) r!} \\
&+ c_1 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+2) r!} \\
&- c_0 \sum_{k=0}^{\infty} \frac{(2)^k t^{\frac{3}{2}k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1) t^{\frac{r}{2}}}{\Gamma(\frac{r}{2}+\frac{3}{2}k+\frac{3}{2}) r!}.
\end{aligned} \tag{8}
$$

Therefore, the two solution equations (3.6) and (3.7) are identical

3.1 Remark

Let 1 < a <2 and a, b ∈ R. Then, the FDE

$$y^{(\alpha)}(t) + ay'(t) + by(t) = 0, \quad t \geq 0.$$

If a = 0 then, the FDE

$$y^{(\alpha)}(t) + by(t) = 0$$

with the initial conditions y(0) = c_0 and y'(0) = c_1 has the following solution

$$y(t) = c_0 \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+1)} + c_1 t \sum_{k=0}^{\infty} \frac{(-bt^{\alpha})^k}{\Gamma(\alpha k+2)}.$$

Theorem 3.3. Let 1 < a < 2 and a, b ∈ R. Then, the FDE

$$y^{(\alpha)}(t) + ay''(t) + by(t) = 0, \quad t \geq 0. \tag{9}$$

with the initial conditions y(0) = c_0 and y'(0)= c_1 has the following solution

$$
\begin{aligned}
y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1) r!} \\
&+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2) r!} \\
&+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3) r!} \\
&+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4) r!}.
\end{aligned}
$$

Proof. Taking Sumudu transform for Equation (9)

$$\frac{G(u)-G(u)}{u^2}-\frac{G'(0)}{u}-\frac{aG(u)-aG(0)}{u^\alpha}-\frac{G'(0)}{u^{\alpha-2}}+bG(u)$$

$$G(u)(u^{-2}+au^{-\alpha}+b)=c_0u^{-2}+c_1u^{-1}+ac_0u^{-\alpha}+ac_1u^{2-\alpha}.$$

Since,

$$\begin{aligned}
\frac{1}{u^{-2}+au^{-\alpha}+b} &= \frac{u^\alpha}{u^{\alpha-2}+a+bu^\alpha},\\
&= \frac{u^\alpha}{(u^{\alpha-2}+a)(1+\frac{bu^\alpha}{u^{\alpha-a}})},\\
&= \frac{u^\alpha}{(u^{\alpha-2}+a)}\sum_{k=0}^{\infty}(\frac{-bu^\alpha}{u^{\alpha-2}+a})^k,\\
&= \sum_{k=0}^{\infty}\frac{(-b)^k u^{k\alpha+\alpha}}{(u^{\alpha-2}+a)^{k+1}},\\
&= \sum_{k=0}^{\infty}\frac{(-b)^k u^{2k+2}}{(1+au^{2-\alpha})^{k+1}},\\
&= \sum_{k=0}^{\infty}(-b)^k u^{2k+2}\sum_{r=0}^{\infty}(-au^{2-\alpha})^r\binom{k+r}{r},\\
&= \sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k+2}.
\end{aligned}$$

Then

$$\begin{aligned}
G(u) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k}\\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k+1}\\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+2}\\
&+ ac_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r u^{(2-\alpha)r+2k-\alpha+3}.
\end{aligned} \tag{10}$$

Taking the inverse Sumudu transform for equation (10).

$$\begin{aligned}
y(t) &= c_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k}}{((2-\alpha)r+2k)!}\\
&+ c_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k+1}}{((2-\alpha)r+2k+1)!}\\
&+ ac_0\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k-\alpha+2}}{((2-\alpha)r+2k-\alpha+2)!}\\
&+ ac_1\sum_{k=0}^{\infty}(-b)^k\sum_{r=0}^{\infty}\binom{k+r}{r}(-a)^r\frac{t^{(2-\alpha)r+2k-\alpha+3}}{((2-\alpha)r+2k-\alpha+3)!}.
\end{aligned}$$

Hence,

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k+2)r!} \\ &+ ac_0 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+2}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+3)r!} \\ &+ ac_1 \sum_{k=0}^{\infty} \frac{(-b)^k t^{2k-\alpha+3}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-at^{2-\alpha})^r}{\Gamma((2-\alpha)r+2k-\alpha+4)r!}. \end{aligned}$$

Example 3.2. If we let $\alpha = \frac{3}{2}$, $a = \sqrt{3}$ and b = 8 in Theorem 3.2, then, the FDE

$$y''(t) + ay^{(\alpha)} + by(t) = 0, \quad t \geq 0. \tag{11}$$

with the initial conditions $y(0) := c_0$ and $y'(0) = c_1$ has the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{12}$$

[18] solved equation (11) using the Laplace transform with the following solution

$$\begin{aligned} y(t) &= c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+1)r!} \\ &+ c_1 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+1}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+2)r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{1}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{3}{2})r!} \\ &+ \sqrt{3}c_0 \sum_{k=0}^{\infty} \frac{(-8)^k t^{2k+\frac{3}{2}}}{k!} \sum_{r=0}^{\infty} \frac{\Gamma(k+r+1)(-\sqrt{3}t^{\frac{1}{2}})^r}{\Gamma(\frac{r}{2}+2k+\frac{5}{2})r!}. \end{aligned} \tag{13}$$

Therefore, the two solutions of Equations (12) and (13) are identical

CONCLUSION

In this study, Fractional Sumudu transform is studied. The operational formulas of the fractional Sumudu transform are derived. Fractional Sumudu transforms and expansions of binomial series coefficients are used to solve some classes of FDEs, leading to several interesting results. The approximate solutions of the problems using the new method agree very well with the analytical solutions. Hence, this method is more economical in terms of computation steps than other existing transformation methods. Therefore, we can conclude that the new method is computationally very efficient for solving some classes of FDEs.

REFERENCES

1. Watugala G., Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Integrated Education, 24 (1993), 35-43.
2. Weerakoon S., Application of Sumudu transform to partial differential equations, International Journal of Mathematical Education in Science and Technology, 25 (1994), 277-283.
3. Watugala G., The Sumudu transform for functions of two variables, Mathematical Engineering in Industry, 8 (2002), 293-302.
4. Belgacem F. B. M.. Karaballi A. A, S. L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical problems in Engineering, 2003, 103–118.
5. Mechee M., Senu N., Numerical study of fractional differential equations of Laneemden type by method of collocation, Applied Mathematics, 3 (2012), 851-856.
6. Mechee M. S. and Senu N., Numerical study of fractional differential equations of laneemden type by the least square method, International Journal of Differential Equations and Applications, 11 (2012), 259-282, 2012.
7. Belgacem F. B. M., Karaballi A. A., Sumudu transform fundamental properties investigations and applications, International Journal of Stochastic Analysis, 2006, 2006.
8. Katatbeh Q. D. and Belgacem F. B. M., Applications of the Sumudu trans- form to fractional differential equations, Nonlinear Studies, 18 (2011), 99-112.
9. Kumar M., Daftardar-Gejji V., Exact solutions of fractional partial differential equations by Sumudu transform iterative method, arXiv preprint arXiv:1806.03057, 2018.
10. Darzi R., Mohammadzade B., Mousavi S., Beheshti R., Sumudu transform method for solving fractional differential equations and fractional diffusion- wave equation, SJ. Math. Comput. Sci, 6 (2013), 79-84.
11. Goswami P., Alqahtani R. T., Solutions of fractional differential equations by Sumudu transform and variational iteration method, Journal of Nonlinear Science and its Applications, 9 (2016), 1944-1951.
12. Bulut H. M., Hasan Baskonus, Belgacem F. B. M., The analytical solution of some fractional ordinary differential equations by the Sumudu transform method, An Abstract and Applied Analysis, Hindawi, 2013.
13. Srivastava M.M., Golmankhaneh A. K., Baleanu D., Yang X.-J., Local fractional Sumudu transform with application to ivps on cantor sets, in Abstract and applied analysis, Hindawi, 2014.
14. Al-Omari S. K. Q., Agarwal P., Some general properties of a fractional Sumudu transform in the class of boehmians, Kuwait Journal of Science, 43 (2016), 2016.
15. Rathore S., Kumar D., Singh R., Gupta S., Homotopy analysis Sumudu transform method for nonlinear equations, International Journal of Indus- trial Mathematics, 4 (2012), 301-314.
16. Gupta V.G., Shrama B., Kiliçman A., A note on fractional Sumudu trans- form, Journal of Applied Mathematics, 2010.
17. Kadem A., Kiliçman A., Note on transport equation and fractional Sumudu transform, S. Computers & Mathematics with Applications, 62 (2011), 2995- 3003.
18. Daniel Zwillinger (12 May 2014). Handbook of Differential Equations. Elsevier Science. ISBN 978-1-4832-2096-3.
19. Lin S.-D., Lu C.-H., Laplace transform for solving some families of fractional differential equations and its applications, Advances in Difference Equations, p. 137, 2013.

20. F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical Problems in Engineering 2003 (2003), no. 3, 103–118.
21. Sumudu transform, 'A new integral transform to solve differential equations and control engineering problems', Mathematical Engineering in Industry 6 (1998), No- 4, 319–329.
22. Kiliman K., Eltayeb H., Agarwal R. P., On Sumudu transform and system of differential equations, in Abstract and Applied Analysis, Hindawi, 2010. Accepted: 22.07.2019.
23. G. M. Mittag-Leffler, "Une generalisation de l'integrale de Laplace-Abel," Comptes Rendus de l'Academie des Sciences S´erie II´, Vol-137, pp. 537–539, 1903.
24. Miller K.S. and Ross B.; An Introduction to the Fractional calculus and Fractional differential Equations, John Wiley and Sons Inc., New York, (1993).
25. Nonnenmacher, T.F., Glocke W.G.; A fractional Model for Mechanical Stress Relaxation, Philosophical Magazine Letters, Vol. 64, No. 2, (1991) 89-93.
26. Abramowitz, M. and Stegun, I. A. (Eds). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1972.
27. Boros, G. and Moll, V. Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge, England: Cambridge University Press, 2004.